BIFURCATIONS OF THE FIXED POINTS OF A POINT TRANSFORMATION UNDER WHICH A ROOT OF THE CHARACTERISTIC POLYNOMIAL PASSES THROUGH THE VALUE $\lambda = -1$

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L.A. KOMRAZ
(Gor'kii)

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The problem of the bifurcations of the fixed points of a point transformation under which a root of the characteristic polynomial passes through the value $\lambda = -1$ (through the surface N_{-1} [1]) involves computing the quantity g_0 whose sign determines the character of the bifurcation. The sign of g_0, however, does not characterize sufficiently fully the behavior of the system near those points of the surface at which g_0 vanishes. The behavior of the system near such points depends essentially on the sign of the quantity h_0 whose computation requires retention of terms of up to the fifth order, inclusive, in the expansion. There is a certain analogy between the quantities g_0 and h_0 and the Liapunov quantities α_3 and α_5 [2 and 3]. We shall show for in the case of a point transformation T of a straight line into a straight line that either one or two pairs of fixed (stable or unstable) points of the transformation T^2 can exist in the neighborhood of a simple fixed point of the transformation T, depending on the quantities g_0 and h_0 (and on the value of a root of the characteristic polynomial). An example is cited of a system described by a nonlinear third-order differential equation in which this bifurcation occurs.

1. Let us consider the point transformation T of a straight line into a straight line

$$\bar{z} = \lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots \qquad (\lambda \equiv a_1)$$

in the neighborhood of the fixed point $\bar{z} = z = 0$. The transformation T^2 is

$$Z = \lambda^2 z + \lambda\ (1 + \lambda)\ a_2 z^2 + g z^3 + f z^4 + h z^5 + \ldots$$

Here

$$g = \lambda[(1 + \lambda^2) a_3 + 2a_2^2] \equiv g_0 + (\lambda^2 - 1) b_3$$
$$f = \lambda\ (1 + \lambda^3) a_4 + \lambda\ (2 + 3\lambda)\ a_2 a_3 + a_2^3 \equiv -{}^1/{}_2 a_2 g_0 + (\lambda^2 - 1) b_4 \qquad (1.1)$$
$$h = \lambda\ (1 + \lambda^4) a_5 + (2 + 3\lambda) a_2^2 a_3 + 2\lambda\ (1 + 2\lambda^2) a_2 a_4 + 3\lambda^2 a_3^2 \equiv h_0 + (\lambda^2 - 1) b_5$$
$$g_0 = -2\ (a_3 + a_2^2), \qquad h_0 = 3a_3^2 - a_2^2 a_3 - 6a_2 a_4 - 2a_5$$

Let us consider the function

$$Z - z \equiv (\lambda^2 - 1) z + \lambda\ (1 + \lambda) a_2 z^2 + g z^3 + f z^4 + h z^5 + \ldots \qquad (1.2)$$

whose zeros correspond to the fixed points of the transformation T^2. Making use of (1.1), we can rewrite function (1.2) as

$$Z - z \equiv (\lambda^2 - 1)(1 + z\Psi_1)\ z + g_0\ (1 + z\Psi_3) z^3 + h_0\ (1 + z\Psi_5) z^5$$

$$\Psi_1 = \frac{\lambda a_3}{\lambda - 1} + b_3 z + b_4 z^2 + b_5 z^3, \quad \Psi_3 = -\frac{1}{2} a_2, \quad \Psi_5 = 1 + (\ldots) z$$

From this we find that

$$Z - z \equiv (1 + z\Psi_1) z \left[\lambda^2 - 1 + g_0 \frac{1 + z\Psi_3}{1 + z\Psi_1} z^2 + h_0 \frac{1 + z\Psi_5}{1 + z\Psi_1} z^4\right] \equiv$$
$$\equiv \Psi_1^* z \left[\lambda^2 - 1 + g_0 \Psi_3^* z^2 + h_0 \Psi_5^* z^4\right]$$

where Ψ_j^* ($j = 1, 3, 5$) are series in powers of z beginning with unity. These series converge [4] inside the ε_0-neighborhood $|a_i^* - a_i| < \varepsilon_0$ of the arbitrary point a_i^* ($a_i^* \neq 1$) of the space of coefficients a_i ($i = 1, 2, \ldots, 5$) for all sufficiently small z ($|z| < \delta_0$).

Let $\lambda + 1 = g_0 = 0$, $h_0 \neq 0$ at the point a_i^*, and let us consider the problem of the number of points of function (1.2) for parameters taken from the neighborhood of the point a_i^*. To be specific, let us assume that $h_0 > 0$ (the reasoning for $h_0 < 0$ is analogous).

The zeros of function (1.2) which are distinct from $z = 0$ coincide with the zeros of the function

$$w = \lambda^2 - 1 + \Psi_3^* g_0 z^2 + \Psi_5^* h_0 z^4 \tag{1.3}$$

Let $\epsilon < 1$ be an arbitrary, arbitrarily small positive number. There exists a δ_1 ($\delta_1 < \delta_0$) such that $|\Psi_j^* - 1| < \epsilon$ ($j = 1, 3, 5$) for $|z| < \delta_1$.

a) Let two sign changes occur in the sequence $\lambda^2 - 1, g_0 h_0$ at the point α_i. We consider the functions

$$w_+ = \lambda^2 - 1 + (1 - \varepsilon) g_0 z^2 + (1 + \varepsilon) h_0 z^4$$
$$w_- = \lambda^2 - 1 + (1 + \varepsilon) g_0 z^2 + (1 - \varepsilon) h_0 z^4$$

On fulfillment of the condition

$$(1 - \varepsilon)^2 g_0^2 - 4(1 + \varepsilon) h_0 (\lambda^2 - 1) > 0$$

each of the equations $w_+ = 0$, $w_- = 0$ has four (two positive and two negative) nonzero roots. There clearly exists and ε_1 ($\varepsilon_1 < \varepsilon_0$), such that for $a_i^* - a_i < \varepsilon_1$ ($a_1^* = -1$ the roots of each of Eqs. $w_+ = 0$ and $w_- = 0$ lie in the δ_1-neighborhood of the point $z = 0$.

The inequality $w_- < w < w_+$ holds for all $|z| < \delta_1$, so that function (1.3) (and with it function (1.2)) has two positive and two negative roots, and the transformation T^2 has two pairs of fixed points. On fulfillment of the condition

$$(1 + \varepsilon)^2 g_0^2 - 4(1 - \varepsilon) h_0 (\lambda^2 - 1) < 0$$

function (1.2) does not have real zeros, and the transformation T^2 does not have fixed points in the neighborhood of the point $z = 0$ (i.e. in the neighborhood of the fixed point of the transformation T).

b) Let not more than one sign change occur in the sequence $\lambda^2 - 1$, g_0, h_0 at the point α_i. Introducing the function

$$W_+ = \lambda^2 - 1 + (1 + \varepsilon) g_0 z^2 + (1 + \varepsilon) h_0 z^4 \qquad W_- < w < W_+$$
$$W_- = \lambda^2 - 1 + (1 - \varepsilon) g_0 z^2 + (1 - \varepsilon) h_0 z^4$$

and reasoning as in case (a) above, we conclude that if the sequence $\lambda^2 - 1$, g_0, h_0 contains one sign change (in which case $h_0 > 0$, $\lambda^2 - 1 < 0$), then function (1.2) has one positive and one negative zero, and the transformation T^2 has two fixed points. If the sequence $\lambda^2 - 1$, g_0, h_0 experiences no sign changes, function (1.2) does not have any real roots, and the transformation T^2 does not have fixed points in the δ_1-neighborhood of the fixed point $z = 0$.

2. In order to investigate the stability of the fixed points of the transformation T^2 in

case (a) when four real zeros, $z_{-2} < z_{-1} < 0 < z_1 < z_2$ exist in the δ_1-neighborhood of the point $z_0 = 0$, we rewrite function (1.2) as

$$Z - z = \Psi_1^* \Psi_5^* h_0 (z - z_{-2}) (z - z_{-1}) z (z - z_1) (z - z_2)$$

Then

$$\frac{dZ}{dz} = 1 + h_0 \varphi(z)$$

where

$$\varphi(z_k) = \Psi_1^* \Psi_5^* \frac{(z - z_{-2})(z - z_{-1}) z (z - z_1)(z - z_2)}{z - z_k} \qquad (k = -2, -1, 0, 1, 2)$$

If δ_1 is chosen in such a way that $16|h_0|\delta_1^4 < 1$, then the estimate $|h_0 \varphi(z_k)| < 1$ is valid. The sign of the quantity changes in passing from k to $k+1$, so that the stable and unstable fixed points alternate. In our case ($h_0 > 0$) the fixed points z_{-2}, z_0, and z_2 are unstable, while z_{-1} and z_1 are stable.

In case (b), when two fixed points of the transformation T^2 exist in the δ_1-neighborhood of the point $z_0 = 0$, we can employ similar reasoning to show that both fixed points are unstable for $h_0 > 0$ (the point $z_0 = 0$ is stable in this case).

In the critical case $\lambda + 1 = g_0 = 0$ we have

$$Z^2 - z^2 = 2h_0 z^6 + \dots$$

The sign of this difference for small z is determined by the sign of h_0. For $h_0 > 0$ the fixed point is unstable in the critical case; for $h_0 < 0$ it is stable.

3. Let us consider the parameter plane g_0, $\lambda^2 - 1$; in this plane the ϵ_1-neighborhood of the point α_i^* ($\lambda + 1 = g_0 = 0$ and $h_0 \neq 0$) of the space of coefficients is associated with some neighborhood of the origin on the plane g_0, $\lambda^2 - 1$. Fig. 1 ($h_0 > 0$) and Fig. 2 ($h_0 < 0$) show the decompositions of this neighborhood into domains according to the number of fixed points of the transformation T^2 in the δ_1-neighborhood of the fixed point $z = 0$ ($z = 0$ is a fixed point of the transformation T).

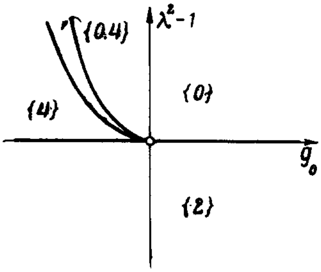

Fig. 1

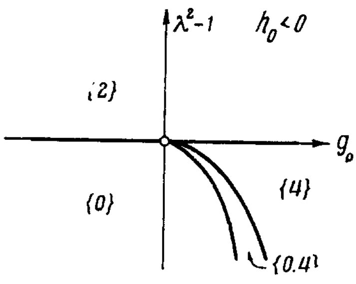

Fig. 2

The transformation T^2 does not have fixed points in the δ_1-neighborhood of the fixed point $z = 0$ in the domain $\{0\}$ isolated by one of the following groups of inequalities:

$$\{h_0 > 0,\ \lambda^2 - 1 \geqslant 0,\ g_0 \geqslant 0\}$$

$$\{h_0 > 0,\ g_0 < 0,\ (1 + \varepsilon)^2 g_0^2 - 4(1 - \varepsilon) h_0 (\lambda^2 - 1) < 0\}$$

$$\{h_0 < 0,\ \lambda^2 - 1 \leqslant 0,\ g_0 \leqslant 0\}$$

$$\{h_0 < 0,\ g_0 > 0,\ (1 + \varepsilon)^2 g_0^2 - 4(1 - \varepsilon) h_0 (\lambda^2 - 1) < 0\}$$

In the domain {2} isolated by one of the groups of inequalities

$$\{h_0 > 0,\ \lambda^2 - 1 < 0,\ g_0 \geqslant 0\} \qquad \{h_0 > 0,\ \lambda^2 - 1 \leqslant 0,\ g_0 < 0\}$$
$$\{h_0 < 0,\ \lambda^2 - 1 > 0,\ g_0 \leqslant 0\} \qquad \{h_0 < 0,\ \lambda^2 - 1 \geqslant 0,\ g_0 > 0\}$$

the transformation T^2 has two fixed points in the δ_1-neighborhood of the fixed point $z = 0$. These points are unstable for $h_0 > 0$ and stable for $h_0 < 0$.

In the domain {4} isolated by the inequalities

$$\{h_0 > 0,\ g_0 < 0,\ (1-\varepsilon)^2 g_0^2 - 4(1+\varepsilon) h_0 (\lambda^2 - 1) > 0\}$$
$$\{h_0 < 0,\ g_0 > 0,\ (1-\varepsilon)^2 g_0^2 - 4(1+\varepsilon) h_0 (\lambda^2 - 1) > 0\}$$

the transformation T^2 has four fixed points in the δ_1-neighborhood of the fixed point $z = 0$. For $h_0 > 0$ the interior pair of fixed points is stable and the outer pair is unstable; the reverse is true in the case $h_0 < 0$.

In the domain {0, 4} isolated by the inequalities

$$\left\{\begin{matrix} h_0 > 0,\ (1-\varepsilon)^2 g_0^2 - 4(1+\varepsilon) h_0 (\lambda^2 - 1) < 0, \\ g_0 < 0,\ (1+\varepsilon)^2 g_0^2 - 4(1-\varepsilon) h_0 (\lambda^2 - 1) > 0 \end{matrix}\right\}$$

or

$$\left\{\begin{matrix} h_0 < 0,\ (1-\varepsilon)^2 g_0^2 - 4(1+\varepsilon) h_0 (\lambda^2 - 1) < 0 \\ g_0 > 0,\ (1+\varepsilon)^2 g_0^2 - 4(1-\varepsilon) h_0 (\lambda^2 - 1) > 0 \end{matrix}\right\}$$

either one of the cases described for the domains {4} and {0} applies to the transformation T^2, or else the transformation T^2 has two semistable fixed points.

4. Example. Let us consider a dynamic system (an electromechanical trigger control) whose motion is described by the equations in dimensionless variables

$$\left.\begin{matrix} x^{\cdot\cdot} + x = -r + y^2 \\ y^{\cdot} + ay = a \end{matrix}\right\} \quad \text{for} \quad x^{\cdot} \geqslant 0,\ |x + b + d| < b \tag{4.1}$$

or

$$\left.\begin{matrix} x^{\cdot\cdot} + x = -r \dfrac{x^{\cdot}}{|x^{\cdot}|} \\ y = 0 \end{matrix}\right\} \quad \begin{matrix} \text{for} \quad x^{\cdot} \geqslant 0,\ |x + b + d| > b \\ \text{or} \quad x^{\cdot} < 0 \end{matrix} \tag{4.2}$$

Transition from (4.2) to (4.1) occurs for $x = -2b - d$, $x^{\cdot} \geqslant 0$, and that from (4.1) to (4.2) for $x = -d$, $x^{\cdot} > 0$. Here a, b, d, and r are parameters which can only be non-negative.

The phase space x, y, $x^{\cdot}$ of the dynamic system under consideration consists of part of the plane and of the three-dimensional domain joined to the latter. Investigation of the breakdown of the phase space into trajectories reduces to the study of the point transformation T of the half-line Γ_1 ($x = -2b - d$, $y = 0$, $\dot{x} \geqslant 0$) into itself. Analysis shows that for a fixed point of the transformation T in the parameter space a, b, d, r there exists a bifurcation surface N_{-1} on which the quantity g_0 changes sign. The quantity g_0 vanishes, for example, for the section $a = 2$, $d = 0.2$ of the surface N_{-1} at the point $b = b_0 \approx 0.164$, $r = r_0 \approx 0.058$. For $b < b_0$ we have $g_0 < 0$; for $b > b_0$ we have $g_0 > 0$. The bifurcation corresponding to the case $h_0 < 0$ occurs in this case.

In this section in the neighborhood of the point (b_0, r_0) of the parameter plane there exists a domain whose points correspond to a transformation T^2 having two stable (outer) and two unstable (inner) fixed points, and also a domain whose points correspond to a transformation T^2 having two stable fixed points. The first of the above domains is associated with a phase space containing a simple stable limiting cycle and two double limiting cycles (an unstable inner cycle and a stable outer cycle). The second domain is associated with a phase space containing a simple unstable and a double stable limiting cycle.

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